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Painlevé V and the distribution function of a discontinuous linear statistic in the Laguerre unitary ensembles

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Abstract
In this paper we study the characteristic or generating function of a certain discontinuous linear statistic of the Laguerre unitary ensembles and show that this is a particular fifth Painlevé transcendent in the variable t, the position of the discontinuity. The proof of the ladder operators adapted to orthogonal polynomial with discontinuous weight announced sometime ago [13] is presented here, followed by the nonlinear difference equations satisfied by two auxiliary quantities and the derivation of the Painlevé equation.

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1. Introduction

In the theory of random matrix ensembles with unitary symmetry, the real eigenvalues $\{x_j\}_{j=1}^n$ have the joint probability distribution

$$\mathsf{P}(x_1,\ldots,x_n)\,\mathrm{d}x_1\cdots\mathrm{d}x_n=\frac{1}{n!D_n}\prod_{1\leqslant j<k\leqslant n}(x_k-x_j)^2\prod_{l=1}^n w_0(x_l)\,\mathrm{d}x_l, \tag{1.1}$$

where $w_0(x)$ with $x\in[a,b]$ say, is strictly positive and satisfies a Lipshitz condition and has finite moments, that is, the existence of the integrals,

$$\int_a^b x^j w_0(x)\,\mathrm{d}x, \qquad j\in\{0,1,2,\ldots\}.$$

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Here D_n is the normalization constant

$$D_n[w_0] = \frac{1}{n!}\int_{[a,b]^n} \prod_{1\leqslant j<k\leqslant n}(x_k - x_j)^2 \prod_{l=1}^{n} w_0(x_l)\,\mathrm{d}x_l, \tag{1.2}$$

so that

$$\int_{[a,b]^n} \mathsf{P}(x_1,\ldots,x_n)\,\mathrm{d}x_1\cdots\mathrm{d}x_n = 1. \tag{1.3}$$

We include the cases where a may be $-\infty$ and/or b may be ∞. For a comprehensive study of the theory of random matrices, see [20].

A linear statistics is a linear sum of a certain function g of the random variable x_j:

$$\sum_{k=1}^{n} g(x_k). \tag{1.4}$$

The generating function of such a linear statistics is, by definition the average of $\exp\left(\lambda\sum_k g(x_k)\right)$, with respect to the joint probability distribution (1.1), where λ is a parameter, reads

$$\int_{[a,b]^n} \exp\left[\lambda\sum_{k=1}^{n} g(x_k)\right]\mathsf{P}(x_1,\ldots,x_n)\prod_{k=1}^{n}\mathrm{d}x_k. \tag{1.5}$$

More generally, we can consider

$$\int_{[a,b]^n}\left[\prod_{k=1}^{n} f(x_k)\right]\mathsf{P}(x_1,\ldots,x_n)\prod_{k=1}^{n}\mathrm{d}x_k. \tag{1.6}$$

If f is a 'smooth' function then asymptotic formulas for large n for the characteristic functions have been obtained for the Hermite case, where $w_0(x) = \mathrm{e}^{-x^2}, x \in (-\infty,\infty)$ by Kac [16] and Akhiezer [1] and generalized by many authors. See [8] for a history of this problem. These results are continuous analogs of the classical Szegö limit theorem on Toeplitz determinants.

In the Laguerre case where $w_0(x) = x^{\alpha}\,\mathrm{e}^{-x}, x \in [0,\infty)$ an analogous formula was found recently for smooth functions in [4]. However, results for the situations where f has discontinuities are harder to come by. We mention here the original studies in [6] where f has several discontinuities and which corresponds to the Hermite case. More general results can be found in [7, 9, 10, 21]. Also, results that correspond to $\alpha = \pm 1/2$ and large n appear in [5].

In this paper we investigate the case where n is finite, and f is constant except for a jump at $t \in [0,\infty)$, and is of the form

$$f(x,t) = A + B\theta(x-t), \tag{1.7}$$

where $\theta(x)$ is 1 for $x > 0$ and 0 otherwise and $A \geqslant 0$ and $B > 0$. In the special case of linear statistics the function g will take the form

$$g(x,t) := \theta(x-t)\ln\left(1+\frac{\beta}{2}\right) + \theta(t-x)\ln\left(1-\frac{\beta}{2}\right), \tag{1.8}$$

where $-1 < \frac{\beta}{2} < 1$. This corresponds to a function f where

$$f(x,t) = \left(1-\frac{\beta}{2}\right)^{\lambda} + \left[\left(1+\frac{\beta}{2}\right)^{\lambda} - \left(1-\frac{\beta}{2}\right)^{\lambda}\right]\theta(x-t)$$

that is

$$A = \left(1-\frac{\beta}{2}\right)^{\lambda}$$

and

$$B = \left(1 + \frac{\beta}{2}\right)^{\lambda} - \left(1 - \frac{\beta}{2}\right)^{\lambda}.$$

We also point out that if $A = 0$ and $B = 1$ then we have the important case where we are computing the probability that all the eigenvalues are in the interval $[t, \infty)$. This case of course is what leads to the now well-known Tracy–Widom laws. More will be said about this later.

Our main tool will be to use the theory of orthogonal polynomials. Previously, in random matrix theory one made use of the orthogonal polynomials associated with the weight that defined the ensemble. Fundamental quantities were then described in terms of Fredholm determinants. While both the authors are very fond of determinants, in this work, we do not consider Fredholm determinants. Instead we consider the polynomials that are orthogonal to the perturbed weight, that is a regular or 'nice' weight multiplied by the discontinuous factor given in (1.7). In this manner we are able to use the results of the orthogonal polynomials to derive equations associated with the various statistics of interest.

The idea is that we write the multiple integral in (1.6) as a Hankel determinant. We then need to know information about the norms of the orthogonal polynomials. To understand this we need to know something about the recursion coefficients of the polynomials. This will lead us naturally to another pair of auxiliary quantities that depend on t and n. In this paper they are called r_n and R_n. Using these auxiliary quantities we are able to produce the second-order nonlinear differential equations satisfied by $S_n = 1 - 1/R_n$ which turns out to be a particular fifth Painlevé transcendent, in addition to the Jimo–Miwa–Okamoto σ form [15] satisfied by the logarithmic derivative of the Hankel determinant with respect to t. We also wish to emphasize that the logarithmic derivative of the Hankel determinant can be computed very naturally in terms of our quantity $r_n(t)$ and its derivative and the relationships between these quantities arise naturally using this approach.

We also derive a discrete version of the σ form of a nonlinear second-order difference equation satisfied by the same logarithmic derivative. Our computations show that in fact the values of our generalized polynomials at the end points of the intervals are intimately related to the resolvent kernels found in the standard approach of Tracy and Widom. This is really not surprising, since we are all starting with the same multiple integral. Rather, our point is that computations can all be made by using only the very basic theory of orthogonal polynomials.

The Painlevé equation can be found in [27]. The second-order difference equation (4.29, theorem 8), as far as we know is a new equation.

In the following section the proof for a pair of ladder operators, and the associated supplementary conditions adapted to orthogonal polynomials with discontinuous weights which was announced sometime ago [13] will be provided. In section 3, a system of difference equations satisfied by two auxiliary quantities r_n and R_n (these will ultimately determine the recurrence coefficients for the orthogonal polynomials) are derived. In section 4 we derive a second-order nonlinear differential equation which turns out to be a particular fifth Painlevé transcendent. In the process we identity that the quantity

$$S_n(t) := 1 - \frac{1}{R_n(t)}$$

to be such an equation. Furthermore we show that the logarithmic derivative of the generating function

$$H_n(t) := t\frac{\mathrm{d}}{\mathrm{d}t}\ln \mathsf{G}(n, t) = t\frac{\mathrm{d}}{\mathrm{d}t}\ln D_n(t)$$

satisfies both the continuous and discrete σ form of Painlevé V.

2. Ladder operators and supplementary conditions

According to the general theory of orthogonal polynomials of one variable, for a generic weight w, the normalization constant (1.2) has the two more alternative representations

$$D_n[w]: = \det(\mu_{i+j})_{i,j=0}^{n-1} := \det\left(\int_a^b x^{i+j} w(x)\,\mathrm{d}x\right)_{i,j=0}^{n-1} \tag{2.1}$$

$$= \prod_{j=0}^{n-1} h_j, \tag{2.2}$$

where the determinant of the moment matrix (μ_{i+j}) is the Hankel determinant. Here, $\{h_j\}_{j=0}^{n}$ is the square of the L^2 norm of the sequence of (monic-)polynomials $\{P_j(x)\}_{j=0}^{n}$ orthogonal with respect to w over $[a, b]$:

$$\int_a^b P_i(x)P_j(x)w(x)\,\mathrm{d}x = \delta_{i,j}h_j. \tag{2.3}$$

Therefore, with reference to (1.2) and (1.5) the quantity that we need to compute is

$$\mathsf{G}(t,n) = \frac{D_n\,[w]}{D_n[w_0]} = \frac{\prod_{i=0}^{n-1} h_i(t)}{\prod_{i=0}^{n-1} h_i},$$

where $w(x, t) := x^\alpha\,\mathrm{e}^{-x}(A + B\theta(x - t))$, and $h_k(t)$ is defined by

$$\int_0^\infty \{P_k(x)\}^2(A + B\theta(x - t))x^\alpha\,\mathrm{e}^{-x}\,\mathrm{d}x = h_k(t). \tag{2.4}$$

We also denote

$$D_n(t) := D_n[w(.,t)].$$

This leads to the generic problem of the characterization of polynomials orthogonal with respect to 'smooth' weights $w_0(x)$ perturbed by a jump factor where the discontinuity is at t. So if we write

$$w_J(x,t) := A + B\theta(x - t), \qquad A \geqslant 0, \quad A + B > 0 \tag{2.5}$$

then

$$\int_a^b P_i(x)P_j(x)w_0(x)w_J(x,t)\,\mathrm{d}x = \delta_{i,j}h_j(t). \tag{2.6}$$

It follows from the orthogonality relations that

$$zP_n(z) = P_{n+1}(z) + \alpha_n(t)P_n(z) + \beta_n(t)P_{n-1}(z). \tag{2.7}$$

This three-term recurrence relation, together with the 'initial' conditions,

$$P_0(z) = 1, \qquad \beta_0 P_{-1}(z) = 0,$$

generates the monic polynomials,

$$P_n(z) = z^n + \mathsf{p}_1(n,t)z^{n-1} + \cdots, \tag{2.8}$$

the first two of which are

$$\begin{aligned} &P_0(z) = 1, \\ &P_1(z) = z - \alpha_0(t) = z - \frac{\mu_1(t)}{\mu_0(t)}. \end{aligned} \tag{2.9}$$

Note that due to the t dependence of the weight, the coefficients of the polynomials and the recurrence coefficients α_n and β_n also depend on t, the position of the jump. However, unless it is required we do not display the t dependence.

From (2.7) and (2.8), we find, for $n \in \{0, 1, 2, \ldots\}$

$$\alpha_n = \mathsf{p}_1(n,t) - \mathsf{p}_1(n+1,t), \qquad \sum_{j=0}^{n-1} \alpha_j = -\mathsf{p}_1(n,t), \tag{2.10}$$

where $\mathsf{p}_1(0,t) := 0$.

From (2.6) and (2.7) we have the well-known strictly positive expression

$$\beta_n := \frac{h_n}{h_{n-1}}. \tag{2.11}$$

Another consequence of the recurrence relation is the Christoffel–Darboux formula

$$\sum_{k=0}^{n-1} \frac{P_k(x)P_k(y)}{h_k} = \frac{P_n(x)P_{n-1}(y) - P_n(y)P_{n-1}(x)}{h_{n-1}(x-y)}. \tag{C − D}$$

The above basic information about orthogonal polynomials can be found in [26].

In this section, we give an account of a recursive algorithm for the determination of the α_n, β_n for a given weight. This is based on a pair of ladder operators and the associated supplementary conditions to be denoted as (S_1) and (S_2). For a general 'smooth' weight the lowering and raising operators have been derived by many authors [3, 11, 12, 24]. We should like to note here Magnus's contribution to this formalism [17–19]. Indeed, we have been motivated by the investigation of [19] where he obtained the large n behavior of the recurrence coefficients of a generalization of the Jacobi polynomials in which the standard Jacobi weight is perturbed by a 'line' analog to the Fisher–Hartwig singularity. We end the discussion about the ladder operators with the remark that the supplementary conditions for orthogonal polynomials on the unit circle was found in [2] and have been used to compute explicitly the Toeplitz determinants with Fisher–Hartwig symbols.

The following lemma gives a detailed proof of the ladder operators in the discontinuous case where the results were announced sometime ago [13].

Lemma 1. *Let $w_0(x), x \in [a,b]$ be a smooth weight function where the associated moments*

$$\int_a^b x^j w_0(x)\,\mathrm{d}x, \qquad j \in \{0, 1, 2, \ldots\} \tag{2.12}$$

of all order exist.

Let $w_0(a) = w_0(b) = 0$, and $\mathsf{v}_0(x) := -\ln w_0(x)$.

The lowering and raising operators for polynomials orthogonal with respect to

$$w(x) := w_0(x) w_J(x,t)$$

are

$$P_n'(z) = -B_n(z)P_n(z) + \beta_n A_n(z) P_{n-1}(z), \tag{2.13}$$

$$P_{n-1}'(z) = [B_n(z) + \mathsf{v}_0'(z)]P_{n-1}(z) - A_{n-1}(z)P_n(z), \tag{2.14}$$

where

$$A_n(z) := \frac{R_n(t)}{z-t} + \frac{1}{h_n}\int_a^b \frac{\mathsf{v}_0'(z) - \mathsf{v}_0'(y)}{z-y} P_n^2(y) w(y)\,\mathrm{d}y \tag{2.15}$$

$$B_n(z) := \frac{r_n(t)}{z-t} + \frac{1}{h_{n-1}} \int_a^b \frac{\mathsf{v}_0'(z) - \mathsf{v}_0'(y)}{z-y} P_n(y) P_{n-1}(y) w(y)\, \mathrm{d}y \tag{2.16}$$

$$R_n(t) := B \frac{w_0(t)}{h_n(t)} \{P_n(t,t)\}^2 \tag{2.17}$$

$$r_n(t) := B \frac{w_0(t)}{h_{n-1}(t)} P_n(t,t) P_{n-1}(t,t), \tag{2.18}$$

where

$$P_n(t,t) := P_n(z,t)|_{z=t}.$$

Here, $\ln w_0(x)$ *is well defined since* $w_0(x)$ *is supposed to be strictly positive for* $x \in [a,b]$.

Proof. We start from

$$P_n'(z) = \sum_{k=0}^{n-1} C_{nk} P_k(z),$$

where C_{nk} is determined from the orthogonality relations

$$C_{nk} = \frac{1}{h_k} \int_a^b P_n'(y) P_k(y) w(y)\, \mathrm{d}y.$$

Therefore,

$$\begin{aligned}
P_n'(z) &= \sum_{k=0}^{n-1} \frac{P_k(z)}{h_k} \int_a^b P_n'(y) P_k(y) w(y)\, \mathrm{d}y \\
&= -\sum_{k=0}^{n-1} \int_a^b \frac{P_k(z)}{h_k} P_n(y) \{P_k'(y) w(y) + P_k(y)[B\delta(y-t) w_0(y) + w_0'(y) w_J(y,t)]\}\, \mathrm{d}y \\
&= -\int_a^b P_n(y) \sum_{k=0}^{n-1} \frac{P_k(z) P_k(y)}{h_k} \left[B w_0(y) \delta(y-t) + \frac{w_0'(y)}{w_0(y)} w(y) \right] \mathrm{d}y \\
&= -\int_a^b P_n(y) \sum_{k=0}^{n-1} \frac{P_k(z) P_k(y)}{h_k} \{B w_0(y) \delta(y-t) + [\mathsf{v}_0'(z) - \mathsf{v}_0'(y)] w(y)\}\, \mathrm{d}y \\
&= -\int_a^b P_n(y) \frac{P_n(z) P_{n-1}(y) - P_n(y) P_{n-1}(z)}{h_{n-1}(z-y)} \{B\delta(y-t) w_0(y) \\
&\qquad + [\mathsf{v}_0'(z) - \mathsf{v}_0'(y)] w(y)\}\, \mathrm{d}y,
\end{aligned}$$

where we have used integration by parts, (C-D), the definition of v_0, (2.11) and that

$$\int_a^b P_n(y) P_k(y) w(y)\, \mathrm{d}y = 0, \qquad k = 0, 1, 2, \ldots, n-1,$$

to arrive at the above. A little simplification produces (2.15), and (2.16) follows from the straightforward application of the recurrence relations. □

Remark 1. If $w_0(a) \neq 0$, $w_0(b) \neq 0$, the terms

$$w(y) \frac{\{P_n(y,t)\}^2}{h_n(t)(z-y)} \Big|_{y=a}^{b} \qquad \text{and} \qquad w(y) \frac{P_n(y,t) P_{n-1}(y,t)}{h_{n-1}(t)(z-y)} \Big|_{y=a}^{b}$$

are to be added into the definition of $A_n(z)$ and $B_n(z)$, respectively.

Remark 2. If there are several jumps at $t_1, \ldots, t_N$ then the first term of (2.15) and (2.16) should be replaced by

$$\sum_{j=1}^{N} \frac{R_{n,j}(t_j; t)}{z - t_j} \qquad \sum_{j=1}^{N} \frac{r_{n,j}(t_j; t)}{z - t_j},$$

where

$$R_{n,j}(t_j; t) := B_j \frac{w_0(t_j)}{h_n(t)} \{P_n(t_j; t)\}^2$$
$$r_{n,j}(t_j; t) := B_j \frac{w_0(t_j)}{h_{n-1}(t)} P_n(t_j; t) P_{n-1}(t_j; t)$$
$$t := (t_1, \ldots, t_N).$$

As in the case of the smooth weight the 'coefficients' $A_n(z)$ and $B_n(z)$ that appear in the ladder operators satisfy two identities valid for all $z \in \mathbb{C} \cup \{\infty\}$, which we gather in the following lemma.

Lemma 2. *The functions $A_n(z)$ and $B_n(z)$ satisfy the following identities which hold for all z:*

$$B_{n+1}(z) + B_n(z) = (z - \alpha_n) A_n(z) - \mathrm{v}_0'(z) \qquad (S_1)$$
$$1 + (z - \alpha_n)[B_{n+1}(z) - B_n(z)] = \beta_{n+1} A_{n+1}(z) - \beta_n A_{n-1}(z). \qquad (S_2)$$

Proof. By a direct computation using the definition of $A_n(z)$ and $B_n(z)$. □

It turns out that a suitable combination of (S_1) and (S_2) produces an identity involving $\sum_{j=0}^{n-1} A_j(z)$, from which further insight into the recurrence coefficients may be gained.

Lemma 3. *$A_n(z)$, $B_n(z)$ and $\sum_{j=0}^{n-1} A_j(z)$ satisfy the identity*

$$[B_n(z)]^2 + \mathrm{v}_0'(z) B_n(z) + \sum_{j=0}^{n-1} A_j(z) = \beta_n A_n(z) A_{n-1}(z). \qquad (S_2')$$

Proof. Multiply (S_2) by $A_n(z)$ and replace $(z - \alpha_n)A_n(z)$ in the resulting equation by $B_{n+1}(z) + B_n(z) + \mathrm{v}_0'(z)$. See (S_1). This leads to

$$[B_{n+1}(z)]^2 - [B_n(z)]^2 + \mathrm{v}_0'(z)[B_{n+1}(z) - B_n(z)] + A_n(z)$$
$$= \beta_{n+1} A_{n+1}(z) A_n(z) - \beta_n A_n(z) A_{n-1}(z).$$

Taking a telescopic sum of the above equation from 0 to $n - 1$ with the 'initial' conditions, $B_0(z) = 0$ and $\beta_0 A_{-1}(z) = 0$, we have (S_2'). □

Let $y = P_n(z)$ we find, by eliminating $P_{n-1}(z)$ from the raising and lowering operators, the second-order differential equation.

Lemma 4.

$$y''(z) - \left(\mathrm{v}_0'(z) + \frac{A_n'(z)}{A_n(z)}\right) y'(z) + \left(B_n'(z) - B_n(z)\frac{A_n'(z)}{A_n(z)} + \sum_{j=0}^{n-1} A_j(z)\right) y(z) = 0. \qquad (2.19)$$

Proof. By a straightforward computation using (2.15), (2.16) and (S_2'). □

Recalling (2.17) and (2.18) we note that if $\mathrm{v}_0'(z)$ is rational in z then the difference kernel, $[\mathrm{v}_0'(z) - \mathrm{v}_0'(y)]/(z - y)$, is rational in z and y. Consequently, (S_1) and (S_2') may be put to good use to obtain a system of difference equations satisfied by the auxiliary quantities R_n and r_n and the recurrence coefficients α_n and β_n. This will be clear in the following section.

3. Recurrence coefficients and difference equations

For the problem at hand,

$$w_0(x) = x^{\alpha}\,\mathrm{e}^{-x}, \qquad x \in [0, \infty), \qquad \mathrm{v}_0(x) := -\ln w_0(x) = -\alpha \ln x + x$$

and for $\alpha > 0$, $w_0(0) = 0$. Note that $w(\infty) = 0$. An easy computation gives

$$\frac{\mathrm{v}_0'(z) - \mathrm{v}_0'(y)}{z - y} = \frac{\alpha}{zy}.$$

Using these and integration by parts we have the following lemma.

Lemma 5.

$$A_n(z) = \frac{R_n(t)}{z - t} + \frac{1 - R_n(t)}{z}, \tag{3.1}$$

$$B_n(z) = \frac{r_n(t)}{z - t} - \frac{n + r_n(t)}{z}, \tag{3.2}$$

where

$$R_n(t) := B w_0(t) \frac{\{P_n(t,t)\}^2}{h_n(t)}, \tag{3.3}$$

$$r_n(t) := B w_0(t) \frac{P_n(t,t) P_{n-1}(t,t)}{h_{n-1}(t)}. \tag{3.4}$$

Proof. Through integration by parts, we find

$$\alpha \int_0^{\infty} y^{\alpha-1}\,\mathrm{e}^{-y} w_J(y;t) \{P_n(y,t)\}^2\,\mathrm{d}y = h_n(t) - B w_0(t) \{P_n(t,t)\}^2 \tag{3.5}$$

$$\alpha \int_0^{\infty} y^{\alpha-1}\,\mathrm{e}^{-y} w_J(y;t) P_n(y,t) P_{n-1}(y,t)\,\mathrm{d}y = -n h_{n-1}(t) - B w_0(t) P_n(t,t) P_{n-1}(t,t), \tag{3.6}$$

and we have used the fact that

$$\frac{\partial}{\partial x} P_n(x,t) = n P_{n-1}(x,t) + \text{lower degree}$$

to arrive at (3.6). From (3.5) and (3.6) and the definitions of $A_n(z)$ and $B_n(z)$, (3.1)–(3.4) follow. □

Substituting (3.1) and (3.2) into (S_1), we find by equating the residues

$$r_{n+1} + r_n = R_n(t - \alpha_n) \tag{3.7}$$

$$-(r_{n+1} + r_n) = 2n + 1 + \alpha - \alpha_n(1 - R_n). \tag{3.8}$$

Lemma 6.

$$\alpha_n = 2n + 1 + \alpha + t R_n, \tag{3.9}$$

$$r_{n+1} + r_n = R_n(t - \alpha_n). \tag{3.10}$$

Proof. Equations (3.7)+(3.8) implies (3.9) and we restate (3.7) as (3.10). □

Substituting (3.1) and (3.2) into (S_2'), we find, after some elementary but messy computations,

$$[B_n(z)]^2 + \mathsf{v}_0'(z)B_n(z) + \sum_{j=0}^{n-1} A_j(z) = \frac{r_n^2}{(z-t)^2} + \frac{(n+r_n)(\alpha+n+r_n)}{z^2} + \frac{\sum_{j=0}^{n-1} R_j + r_n\left[1-\frac{\alpha}{t}-\frac{2(n+r_n)}{t}\right]}{z-t} + \frac{1}{z}\left[n - \sum_{j=0}^{n-1} R_j + (n+r_n)\left(\frac{2r_n}{t}-1\right) + \frac{\alpha r_n}{t}\right] \tag{3.11}$$

and

$$\beta_n A_n(z)A_{n-1}(z) = \frac{\beta_n R_n R_{n-1}}{(z-t)^2} + \frac{\beta_n(1-R_n)(1-R_{n-1})}{z^2} + \frac{1}{t}\left(\frac{1}{z-t}-\frac{1}{z}\right)\beta_n[(1-R_n)R_{n-1} + (1-R_{n-1})R_n]. \tag{3.12}$$

Now (S_2') implies the following lemma.

Lemma 7. *For a fixed t, the quantities r_n, R_n, β_n satisfy the equations*

$$r_n^2 = \beta_n R_n R_{n-1} \tag{3.13}$$

$$(n+r_n)(n+\alpha+r_n) = \beta_n(1-R_n)(1-R_{n-1}) \tag{3.14}$$

$$\sum_{j=0}^{n-1} R_j + r_n\left[1-\frac{\alpha}{t}-\frac{2(n+r_n)}{t}\right] = \frac{\beta_n}{t}\,[(1-R_n)R_{n-1} + (1-R_{n-1})R_n]\,. \tag{3.15}$$

Proof. Equations (3.13)–(3.15) are obtained by equating residues of (S_2'). □

In the following lemma an expression is found for β_n in terms of r_n and R_n.

Lemma 8. *In terms of r_n and R_n, the off-diagonal recurrence coefficient (β_n) reads*

$$\beta_n = \frac{1}{1-R_n}\left[r_n(2n+\alpha) + n(n+\alpha) + \frac{r_n^2}{R_n}\right]. \tag{3.16}$$

Proof. We eliminate $\beta_n R_n R_{n-1}$ from (3.13) and (3.14) to find

$$r_n(2n+\alpha) + n(n+\alpha) = \beta_n(1-R_n-R_{n-1}) \tag{3.17}$$

$$= \beta_n(1-R_n) - \frac{r_n^2}{R_n}. \tag{3.18}$$

In the last step we have used (3.13) to replace $\beta_n R_{n-1}$ by r_n^2/R_n. □

We note that $B > 0$ can always be satisfied for the proper range of λ.

Equation (3.9) states that α_n is linear in R_n up to a linear form in n, together with (3.10) and (3.16), when combined with say, (3.13) provide us with a pair of nonlinear difference equations satisfied by r_n and R_n. We state this in the following theorem.

Theorem 1. *The quantities r_n and R_n satisfy the difference equations*

$$r_{n+1} + r_n = R_n(t - 2n - \alpha - 1 - tR_n) \tag{3.19}$$

$$r_n^2\left(\frac{1}{R_nR_{n-1}} - \frac{1}{R_n} - \frac{1}{R_{n-1}}\right) = r_n(2n+\alpha) + n(n+\alpha) \tag{3.20}$$

with the 'initial' conditions

$$r_0(t) = 0 \tag{3.21}$$

$$R_0(t) = \frac{Bt^\alpha\,\mathrm{e}^{-t}}{h_0(t)} \tag{3.22}$$

$$h_0(t) = \left(1 - \frac{\beta}{2}\right)^\lambda \Gamma(1+\alpha) + \left[\left(1+\frac{\beta}{2}\right)^\lambda - \left(1 - \frac{\beta}{2}\right)^\lambda\right]\int_t^\infty x^\alpha\,\mathrm{e}^{-x}\,\mathrm{d}x. \tag{3.23}$$

Proof. This is simply a restatement of (3.10) and (3.13) with (3.9) and (3.16). □

We shall see that (S_2') automatically performs finite sums in 'local' form, of the quantities R_n and α_n. This will be seen later to be relevant in the evaluation of the derivative of $\ln D_n(t)$ with respect to t and the derivation of the Painlevé transcendent.

Theorem 2.

$$t\sum_{j=0}^{n-1} R_j = -tr_n - n(n+\alpha) + \beta_n \tag{3.24}$$

$$\sum_{j=0}^{n-1}\alpha_j = -\mathrm{p}_1(n) = \beta_n - tr_n. \tag{3.25}$$

Proof. From (3.15), we have

$$\begin{aligned} t\sum_{j=0}^{n-1} R_j &= r_n[2(n+r_n) + \alpha - t] + \beta_n[R_n + R_{n-1} - 2R_nR_{n-1}] \\ &= r_n[2(n+r_n) + \alpha - t] + \beta_n[R_n + R_{n-1}] - 2r_n^2 \\ &= r_n[2(n+r_n) + \alpha - t] + \beta_n - r_n(2n+\alpha) - n(n+\alpha) - 2r_n^2 \\ &= -tr_n - n(n+\alpha) + \beta_n. \end{aligned} \tag{3.26}$$

The second equality of (3.26) follows from (3.13) and the third equality follows from (3.17). Equation (3.25) follows from (3.9) and the second equality of (2.10). □

4. $\mathbf{P}_V\left(0, -\frac{\alpha^2}{2}, 2n+1+\alpha, -\frac{1}{2}\right)$

In this section we shall discover which of the auxiliary quantities defined as the residues of the rational functions $A_n(z)$ and $B_n(z)$ is a Painlevé transcendent.

This will be obtained from a pair of Toda equations which shows that the Hankel determinant is the τ-function and these when suitably combined with the difference equations produce our P_V.

Taking the derivative of $h_n(t)$ with respect to t, we find

$$\frac{\mathrm{d}}{\mathrm{d}t}\ln h_n(t) = -Bw_0(t)\frac{\{P_n(t,t)\}^2}{h_n(t)} = -R_n(t), \tag{4.1}$$

and consequently we have the following theorem.

Theorem 3.

$$\begin{aligned}-t\frac{\mathrm{d}}{\mathrm{d}t}\ln D_n(t) &= -t\sum_{j=0}^{n-1}\frac{\mathrm{d}}{\mathrm{d}t}\ln h_j(t)\\ &= t\sum_{j=0}^{n-1}R_j = -\mathsf{p}_1(n,t) - n(n+\alpha).\end{aligned} \tag{4.2}$$

Proof. The proof is obvious. □

The following lemma gives the derivative of $\mathsf{p}_1(n,t)$ with respect to t.

Lemma 9.

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathsf{p}_1(n,t) = r_n(t). \tag{4.3}$$

Proof. Note the t dependence of $\mathsf{p}_1(n,t)$. Taking a derivative of

$$0 = \int_0^\infty P_n(x)P_{n-1}(x)w_J(x,t)w_0(x)\,\mathrm{d}x,$$

with respect to t, produces

$$\begin{aligned}0 &= -Bw_0(t)P_n(t,t)P_{n-1}(t,t) + \int_0^\infty\left[\frac{\mathrm{d}}{\mathrm{d}t}\mathsf{p}_1(n,t)x^{n-1}+\cdots\right]P_{n-1}(x)w_J(x,t)w_0(x)\,\mathrm{d}x\\ &= -Bw_0(t)P_n(t,t)P_{n-1}(t,t) + h_{n-1}\frac{\mathrm{d}}{\mathrm{d}t}\mathsf{p}_1(n,t)\end{aligned}$$

and (4.3) follows. □

We expect $D_n(t)$ to satisfy the Toda molecule equation [25] and this should indicate the emergence of a Painlevé transcendent. The question that we will address is 'Which quantity is satisfied by this particular Painlevé transcendent?'

Theorem 4. *The Hankel determinant $D_n(t)$ satisfy the following differential-difference or the Toda molecule equation [25]:*

$$t^2\frac{\mathrm{d}^2}{\mathrm{d}t^2}\ln D_n(t) = -n(n+\alpha) + \frac{D_{n+1}(t)D_{n-1}(t)}{D_n^2(t)}. \tag{4.4}$$

Proof. Taking a derivative of (4.2) with respect to t and (4.3) imply

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(t\frac{\mathrm{d}}{\mathrm{d}t}\ln D_n(t)\right) = r_n.$$

Now substitute r_n given above into (3.24) to find

$$\begin{aligned}t\sum_{j=0}^{n-1}R_j &= -t\frac{\mathrm{d}}{\mathrm{d}t}\left[t\frac{\mathrm{d}}{\mathrm{d}t}\ln D_n(t)\right] - n(n+\alpha) + \beta_n.\\ &= -t\frac{\mathrm{d}}{\mathrm{d}t}\ln D_n(t),\end{aligned}$$

where the last equality comes from (4.2). Equation (4.4) follows if we recall

$$\beta_n = \frac{h_n}{h_{n-1}} = \frac{D_{n+1}D_{n-1}}{D_n^2},$$

since $D_n = h_0 \cdots h_{n-1}$. □

We now state a pair of somewhat non-standard Toda equations.

Lemma 10. *The recurrence coefficients* α_n *and* β_n *satisfy for* $n \in \{1, 2, \ldots\}$ *the differential-difference equations*

$$\beta_n'(t) = (R_{n-1} - R_n)\beta_n \qquad (T_1)$$

$$\alpha_n'(t) = r_n - r_{n-1}, \qquad (T_2)$$

with $r_0(t)$ *and* $R_0(t)$ *given by (3.22) and (3.23), respectively.*

Proof. These equations are an immediate consequence of (4.1), (2.11), (4.3) and the first equality (2.10). □

To discover the P_V of our problem. We first state two preliminary lemmas describing the t evolution of r_n and R_n.

Lemma 11. *For a fixed* n, $R_n(t)$ *satisfy the Riccati equation*

$$tR_n' = 2r_n + (2n + \alpha - t + tR_n)R_n. \qquad (4.5)$$

Proof. We begin with (T_2) and replace r_{n+1} by $R_n(t - \alpha_n) - r_n$. See (3.7). This leaves

$$\alpha_n' = 2r_n - (t - \alpha_n)R_n.$$

After eliminating α_n in favor of R_n with (3.9) we have (4.5). □

Lemma 12. *For a fixed* n, $r_n(t)$ *satisfy the Riccati equation*

$$tr_n' = \frac{1 - 2R_n}{R_n(1 - R_n)}(r_n)^2 - (2n + \alpha)\frac{R_n r_n}{1 - R_n} - n(n + \alpha)\frac{R_n}{1 - R_n}. \qquad (4.6)$$

Proof. By equating (3.24) to the last equality of (4.2), we find

$$\mathsf{p}_1(n, t) = tr_n - \beta_n.$$

Taking a derivative of the above equation with respect to t and noting (4.3) we see that

$$tr_n' = \beta_n' = [R_{n-1} - R_n]\beta_n = \frac{r_n^2}{R_n} - \beta_n R_n,$$

and use have been made of (T_2) and (3.13) to obtain the last two equalities. Equation (4.6) follows if we express β_n in terms of r_n and R_n using (3.16). □

The following theorem shows that R_n is up to a linear fractional transformation of a particular P_V.

Theorem 5. *The quantity*

$$S_n(t) := 1 - \frac{1}{R_n(t)} \qquad (4.7)$$

satisfies

$$S_n'' = \frac{3S_n - 1}{2S_n(1 - S_n)}(S_n')^2 - \frac{S_n'}{t} - \frac{\alpha^2}{2}\frac{(S_n - 1)^2}{t^2 S_n} + (2n + 1 + \alpha)\frac{S_n}{t} - \frac{1}{2}\frac{S_n(S_n + 1)}{S_n - 1}, \qquad (4.8)$$

which is $P_V(0, -\alpha^2/2, 2n + 1 + \alpha, -1/2)$.

In terms of the recurrence coefficient $\alpha_n(t)$, *we have*

$$S_n(t) = \frac{\alpha_n(t) - (2n+\alpha+1) - t}{\alpha_n(t) - (2n+\alpha+1)}. \tag{4.9}$$

Proof. Eliminate $r_n(t)$ from (4.5) and (4.6) and with $R_n = 1/(1-S_n)$ gives (4.8). We have followed the convention of [14]. □

Remark 3. Note that for $n = 0$, (4.8) is satisfied by

$$S_0(t) = 1 - \frac{1}{R_0(t)},$$

where $R_0(t)$ is given by (3.22) and (3.23) and ultimately in terms of an incomplete gamma function—a special case of the Kummer function of the second kind. Furthermore, since $r_0(t) = 0$, it can be verified that $R_0(t)$ also satisfy (4.5) at $n = 0$.

We may express the logarithmic derivative of $D_n(t)$ with respect to t in the so-called Jimbo–Miwa–Okamoto σ form. This is described in the following theorem.

Theorem 6. *Let*

$$H_n(t) := t\frac{\mathrm{d}}{\mathrm{d}t}\ln D_n(t), \tag{4.10}$$

then

$$(tH_n'')^2 = 4(H_n')^2[H_n - n(n+\alpha) - tH_n'] + [(2n+\alpha-t)H_n' + H_n]^2. \tag{4.11}$$

Proof. First we express $r_n(t)$ and $\beta_n(t)$ in terms of H_n and its derivatives. From (3.24) and (4.2), we have

$$\begin{aligned} -H_n &= -tr_n + \beta_n - n(n+\alpha) \\ &= -\mathsf{p}_1(n,t) - n(n+\alpha). \end{aligned} \tag{4.12}$$

Taking a derivative of (4.2) with respect to t and recalling (4.3), we have

$$r_n = H_n', \tag{4.13}$$

and with the first equality of (4.12) and (4.13), we find

$$\beta_n = tH_n' - H_n + n(n+\alpha). \tag{4.14}$$

Now a derivative of (4.14) with respect to t and (T_1) gives

$$\begin{aligned} (tH_n')' - H_n' &= tH_n'' \\ &= \beta_n' = (R_{n-1} - R_n)\beta_n \\ &= \frac{r_n^2}{R_n} - \beta_n R_n. \end{aligned} \tag{4.15}$$

Here we have made use of (3.13) to arrive at the last equality. Therefore, we have a quadratic equation in R_n:

$$\frac{r_n^2}{R_n} - \beta_n R_n = tH_n''. \tag{4.16}$$

There is another quadratic equation in R_n which is a restatement of (3.16):

$$\frac{r_n^2}{R_n} + \beta_n R_n = \beta_n - (2n+\alpha)r_n - n(n+\alpha). \tag{4.17}$$

Now we solve for R_n and $1/R_n$ from (4.16) and (4.17) and find

$$\frac{2r_n^2}{R_n} = \beta_n - (2n+\alpha)r_n - n(n+\alpha) + tH_n''$$
$$2\beta_n R_n = \beta_n - (2n+\alpha)r_n - n(n+\alpha) - tH_n''.$$

Equation (4.10) follows from the product of the above two equations,

$$4\beta_n r_n^2 = [\beta_n - (2n+\alpha)r_n - n(n+\alpha)]^2 - (tH_n'')^2,$$

and (4.13) and (4.14). □

Incidentally, R_n has two alternative representations:

$$R_n = \frac{tH_n'' + (2n+\alpha-t)H_n' + H_n}{2[H_n - n(n+\alpha) - tH_n']}, \tag{4.18}$$

$$\frac{1}{R_n} = \frac{tH_n'' - (2n+\alpha-t)H_n' - H_n}{2(H_n')^2}. \tag{4.19}$$

The 'discrete' structure inherited from the recurrence relations (2.7) induces a discrete analog of the σ form, namely, a nonlinear second-order difference equation in n satisfied by H_n for a fixed t; we believe such a discrete form is new and may have been missed in previous similar studies perhaps because the recurrence relations were not sufficiently exploited. We note here that our derivation of (4.11) bypasses a third-order equation and without having to identify a first integral which reduces the order by 1.

We also note that equation (4.11) was first discovered by Tracy and Widom in [27] (which in our problem corresponds to $A = 0$ and $B = 1$) and just as was done in their paper for the Hermite case one can also rescale to obtain the Painlevé III equation corresponding to the Bessel kernel or 'hard edge scaling'. We change variables $t \to s/4n$, $H_n \to \sigma$ use (4.11) and keep only the highest order terms to obtain

$$(s\sigma'')^2 = 4\sigma(\sigma')^2 - 4s(\sigma')^3 - s(\sigma')^2 + \sigma\sigma' + \alpha^2(\sigma')^2.$$

Finally, we point out that the above analysis shows that the resolvent kernel used in the Tracy–Widom approach can be directly related to the orthogonal polynomials defined on (t,∞). In fact, if we denote $\tilde{R}(t,t)$ as the resolvent kernel defined in [27] then

$$t\tilde{R}(t,t) = H_n(t) = -tr_n - n(n+\alpha) + \beta_n.$$

Thus,

$$t\tilde{R}(t,t) = -t\frac{Bw_0 P_n(t,t)P_{n-1}(t,t)}{h_{n-1}(t)} - n(n+\alpha) + \frac{h_n(t)}{h_{n-1}(t)}.$$

The term β_n can also be written using (3.16). In addition, we have that

$$\frac{\mathrm{d}}{\mathrm{d}t}(t\tilde{R}(t,t)) = r_n = \frac{Bw_0 P_n(t,t)P_{n-1}(t,t)}{h_{n-1}(t)}.$$

In other words, we have found an identity for the resolvent kernel in terms of the values at the end points of the normalized orthogonal polynomials.

Theorem 7. *The auxiliary quantities R_n and r_n are expressed in terms of H_n and $H_{n\pm1}$ as follows:*

$$tR_n = H_n - H_{n+1} \tag{4.20}$$

$$tr_n = \frac{[H_n - n(n+\alpha)](t + H_{n+1} - H_{n-1}) + tn(n+\alpha)}{t + H_{n+1} - H_{n-1} - 2n - \alpha}. \tag{4.21}$$

The discrete analog of the σ form satisfied by H_n results from the substitution of (4.20) and (4.21) into

$$(tr_n)^2 = [n(n+\alpha) + tr_n - H_n][(tR_n)^2 + tR_n(H_{n+1} + H_{n-1} - 2H_n)]. \tag{4.22}$$

Proof. Taking a first-order difference on the second equality of (4.2) together with (2.10) and (3.9) implies (4.20). □

We rewrite (3.24) and give

$$\beta_n = n(n+\alpha) + tr_n - H_n. \tag{4.23}$$

We will now find another equation expressing β_n in terms of $r_n, R_n, H_n, H_{n\pm1}$. Taking a first-order difference on (4.20) gives

$$t(R_n - R_{n-1}) = 2H_n - H_{n+1} - H_{n-1}.$$

Now multiply the above equation by R_n and make use of (3.13) we find

$$tR_n^2 - \frac{tr_n^2}{\beta_n} = (2H_n - H_{n+1} - H_{n-1})R_n,$$

and therefore

$$\frac{1}{\beta_n} = \frac{tR_n^2 - (2H_n - H_{n+1} - H_{n-1})R_n}{tr_n^2}. \tag{4.24}$$

Therefore, the product of (4.23) and (4.24) implies (4.21), which leaves us the job of finding a further expression of r_n in terms of H_n and $H_{n\pm1}$. For this purpose, we rewrite (3.17) as

$$\beta_n(1 - R_n - R_{n-1}) = (2n+\alpha)r_n + n(n+\alpha).$$

Now substitute β_n given in (4.23) into the above resulting a linear equation in r_n:

$$r_n[(t - tR_n - tR_{n-1}) - 2n - \alpha] = [H_n - n(n+\alpha)](1 - R_n - R_{n-1}) + n(n+\alpha).$$

With tR_n as in (4.20), we have (4.21).

We summarize our results in the following theorem.

Theorem 8. *Let $D_n(t)$ be the Hankel determinant associated with the Laguerre weight perturbed by a jump factor, and*

$$H_n(t) := t\frac{\mathrm{d}}{\mathrm{d}t}\ln D_n(t).$$

Then the recurrence coefficients are

$$\alpha_n(t) - (2n+\alpha+1) = \frac{t^2H_n'' + [(2n+\alpha)t - t^2]H_n' + tH_n}{2[H_n - n(n+\alpha) - tH_n']}, \tag{4.25}$$

$$\beta_n(t) - n(n+\alpha) = tH_n' - H_n, \tag{4.26}$$

where $2n+1+\alpha$ and $n(n+\alpha)$ are the 'unperturbed' recurrence coefficients and H_n satisfies a nonlinear differential equation in the Jimbo–Miwa–Okamoto σ form

$$(tH_n'')^2 = 4(H_n')^2[H_n - n(n+\alpha) - tH_n'] + [(2n+\alpha-t)H_n' + H_n]^2.$$

For the same H_n, the recurrence coefficients are

$$\alpha_n(t) - (2n + \alpha + 1) = H_n - H_{n+1} \tag{4.27}$$

$$\beta_n(t) - n(n + \alpha) = \frac{H_n(2n + \alpha) - n(n + \alpha)(H_{n+1} - H_{n-1})}{t + H_{n+1} - H_{n-1} - 2n - \alpha}, \tag{4.28}$$

where H_n satisfies the discrete σ form of a nonlinear difference equation

$$\left\{\frac{[H_n - n(n + \alpha)](t + H_{n+1} - H_{n-1}) + tn(n + \alpha)}{t + H_{n+1} - H_{n-1} - 2n - \alpha}\right\}^2 = \left\{\frac{(2n + \alpha)[H_n - n(n + \alpha)] + tn(n + \alpha)}{t + H_{n+1} - H_{n-1} - 2n - \alpha}\right\}(H_n - H_{n+1})(H_{n-1} - H_n). \tag{4.29}$$

Note that since $\alpha_n(t)$ and $\beta_n(t)$ have two alternative representations, $H_n(t)$ satisfies two more differential-difference equations, (4.25) = (4.27) and (4.26) = (4.28).

We end this paper with a discussion on the relationship between our P_V and the difference equations (3.19) and (3.20). We would like to thank the second referee for supplying us the background material part of which is reproduced here.

The fifth Painlevé equation $P_V(a, b, c, d = -1/2)$

$$y'' = \left(\frac{1}{2y} + \frac{1}{y - 1}\right)(y')^2 - \frac{y'}{t} + \frac{(y - 1)^2}{t^2}\left(ay + \frac{b}{y}\right) + c\frac{y}{t} + d\frac{y(y + 1)}{y - 1}, \qquad ' = \frac{\mathrm{d}}{\mathrm{d}t}$$

is equivalent to the Hamiltonian system $\mathcal{H}_V$:

$$q' = \frac{\partial \mathsf{H}}{\partial p}, \qquad tp' = -\frac{\partial \mathsf{H}}{\partial q},$$

with the time-dependent Hamiltonian $\mathsf{H} = \mathsf{H}(p, q, t)$:

$$t\mathsf{H} = p(p + t)q(q - 1) + \alpha_2 qt - \alpha_3 pq - \alpha_1 p(q - 1),$$

where

$$a = \frac{\alpha_1^2}{2}, \qquad b = -\frac{\alpha_3^2}{2}, \qquad c = \alpha_0 - \alpha_2, \qquad d = -\frac{1}{2},$$
$$\alpha_0 := 1 - \alpha_1 - \alpha_2 - \alpha_3$$

and

$$y = 1 - \frac{1}{q}.$$

The Hamiltonian structure was studied in [22] and the τ-function is defined such that

$$\frac{\mathrm{d}}{\mathrm{d}t} \ln \tau = \mathsf{H}.$$

The extended affine Weyl group $W\left(A_3^{(1)}\right) = \langle s_0, s_1, s_2, s_3, \pi\rangle$ of the Weyl group type $A_3^{(1)}$ acts as bi-rational symmetries on P_V and induces Backlund transformations on the solutions of P_V. Here the s_i's and π are the generators. See [22] for the study of Weyl group actions on P_V.

For example, the action of s_0

$$s_0\{\alpha_0, \alpha_1, \alpha_2, \alpha_3\} = \{-\alpha_0, \alpha_1 + \alpha_0, \alpha_2, \alpha_3 + \alpha_0\},$$
$$s_0(q) = q + \frac{\alpha_0}{p + t},$$
$$s_0(p) = p$$

leaves P_V or the Hamiltonian system $\mathcal{H}_V$ invariant. We refer the readers to [22, 28] for information on Weyl group actions and ((4.3), [28]) which lists the bi-rational transformations.

To proceed further, consider a parallel transformation $l = (s_2 s_3 \pi)^2 \in W\big(A_3^{(1)}\big)$:

$$l : \vec{\alpha} = (\alpha_0, \alpha_1, \alpha_2, \alpha_3) \longmapsto \vec{\alpha} + (1, 0, -1, 0).$$

From a direct computation, we may verify that the variables q and $r := pq(q-1)$ satisfy the following system of difference equations:

$$l(r) + r = q(\alpha_2 - \alpha_0 + t - tq) - \alpha_1 \tag{4.30}$$

$$\left(\frac{1}{q} - 1\right)\left(\frac{1}{l^{-1}(q)} - 1\right) = \frac{(r-\alpha_2)(r-\alpha_2-\alpha_3)}{r(r+\alpha_1)}, \tag{4.31}$$

these seems to the $d - P_{\rm III}$ of (205, [23]) in disguise.

In terms of H our auxiliary parameter r reads

$$r = \frac{\mathrm{d}}{\mathrm{d}t}(t\mathsf{H}). \tag{4.32}$$

In our problem we have $P_V(0, -\alpha^2/2, 2n+1+\alpha, -1/2)$, which implies

$$(\alpha_0, \alpha_1, \alpha_2, \alpha_3) = (1+n, 0, -n-\alpha, \alpha).$$

If

$$l^n(q) = q_n =: R_n$$
$$l^n(r) =: r_n,$$

for $n \in \{0, 1, 2, \ldots\}$, then a direct computation shows that

$$r_{n+1} + r_n = R_n(-\alpha - 2n - 1 + t - tR_n), \tag{4.33}$$

$$\left(\frac{1}{R_n} - 1\right)\left(\frac{1}{R_{n-1}} - 1\right) = \frac{(r_n+n+\alpha)(r_n+n)}{r_n^2} \tag{4.34}$$

are equivalent to (3.19) and (3.20), respectively.

We should like to mention here that (3.19) and (3.20) and other equations are derived entirely from orthogonality and the immediate consequence—the recurrence relations.

In view of (3.32) we see that the logarithmic derivative of the generating function $\mathsf{G}(n, t) = D_n(t)$ is the τ-function of our P_V. We end this paper with the final remark: equation (4.4) is essentially the same as the Toda equation among a τ-sequence discovered by Okamoto [22].

References

[1] Akhiezer N I 1964 Continual analogue to some theorems on Toeplitz matrices *Am. Math. Soc. Trans. Ser.* **50** 295–316

[2] Basor E and Chen Y 2008 Toeplitz determinants from compatibility conditions *Ramanujan J.* **16** 25–40

[3] Bonan S and Nevai P 1984 Orthogonal polynomials and their derivatives: I *J. Approx. Theory* **40** 134–47

Bonan S and Clark D S 1986 Estimates of the orthogonal polynomials with weight $\exp(-x^m)$, m and even integer *J. Approx. Theory* **46** 408–10

Bonan S, Lubinsky D S and Nevai P 1987 Orthogonal polynomials and their derivatives: II *J. Approx. SIAM J. Math. Anal.* **18** 1163–76

[4] Basor E L, Chen Y and Widom H 2001 Determinants of Hankel matrices *J. Funct. Anal.* **179** 214–34

[5] Basor E L, Ehrhardt T and Widom H 2003 On the determinant of a certain Wiener–Hopf + Hankel operator *Integral Eqns Operator Theory* **47** 275–88

[6] Basor E L and Widom H 1983 Toeplitz and Wiener–Hopf determinants with piecewise continuous symbols *J. Funct. Anal.* **50** 387–413

[7] Basor E L and Widom H 2005 Wiener–Hopf determinants with Fisher–Hartwig symbols *Operator Theoretical Methods and Applications to Mathematical Physics* (*Oper. Theory Adv. Appl.* vol 147) (Basel: Birkhauser) 131–49

[8] Böttcher A and Silbermann B 1990 *Analysis of Toeplitz Operators* (Berlin: Springer)

[9] Böttcher A, Silbermann B and Widom H 1994 A continuous analogue of the Fisher–Hartwig formula for piecewise continuous symbols *J. Funct. Anal.* **122** 222–46

[10] Böttcher A, Silbermann B and Widom H 1994 Determinants of truncated Wiener–Hopf operators with Hilbert–Schmidt kernels and piecewise continuous symbols *Arch. Math.* **63** 60–71

[11] Chen Y and Ismail M 1997 Ladder operators and differential equations for orthogonal polynomials *J. Phys. A: Math. Gen.* **30** 7817–29

[12] Chen Y and Ismail M 2005 Jacobi polynomials from compatibility conditions *Proc. Am. Math. Soc.* **133** 465–72

[13] Chen Y and Pruessner G 2005 Orthogonal polynomials with discontinuous weights *J. Phys. A: Math. Gen.* **38** L191–8

[14] Gromak V I, Laine I and Shinomura S 2002 *Painlevé Differential Equations in the Complex Plane* (New York: Springer)

[15] Jimbo M and Miwa T 1981 Monodromy preserving deformation of linear ordinary differential equations with rational coefficients: II *Physica* D **2** 407–48

Okamoto K 1982 On the τ-function of the Painlevé equations *Physica* D **2** 525–35

[16] Kac M 1954 Toeplitz matrices, translation kernel and a related problem in probability theory *Duke Math. J.* **21** 501–9

[17] Magnus A P 1995 Painlevé-type differential equations for the recurrence coefficients of semi-classical orthogonal polynomials *J. Comput. Appl. Math.* **57** 215–37

[18] Magnus A P 2006 Freud equations for Legendre polynomials on a circular arc and solution of the Grübaum–Delsarte–Janssen–Vries problem *J. Approx. Theory* **139** 75–90

[19] Magnus A P 1995 Asymptotics for the simplest generalized Jacobi polynomials recurrence coefficients from Freud's equations: numerical exploration *Ann. Numer. Math.* **2** 311–25

[20] Mehta M L 2006 *Random Matrices* 3rd edn (Singapore: Elsevier)

[21] Mikaelyan L V 1986 Asymptotics of determinants of truncated Wiener–Hopf operators in a singular case *Akad. Nauk Arm. SSR Dokl.* **82** 151–5 (in Russian)

[22] Okamoto K 1987 Studies on the Painlevé equations: II. Fifth Painlevé equation P_{V} *Japan. J. Math. (N. S.)* **13** 47–76

[23] Sakai H 2001 Rational surfaces associated with affine root systems and geometry of the Painlevé equations *Commun. Math. Phys.* **220** 165–229

[24] Shohat J 1939 A differential equation for orthogonal polynomials *Duke Math. J.* **5** 401–17

[25] Sogo K 1993 Time dependent orthogonal polynomials and theory of soliton—applications to matrix model, vertex model and level statistics *J. Phys. Soc. Japan* **62** 1887–94

[26] Szegö G 1975 *Orthogonal Polynomials* Fourth ed (*Am. Math. Soc. Colloq. Publ.* vol 23) (Providence, RI: American Mathematical Society)

[27] Tracy C A and Widom H 1994 Fredholm determinants, differential equations and matrix models *Commun. Math. Phys.* **163** 33–72

[28] Tsuda T, Okamoto K and Folding Sakai H 2005 Transformations of the Painlevé equations *Math. Ann.* **331** 713–38